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# APERIODIC TILINGS OF THE HYPERBOLIC PLANE BY CONVEX POLYGONS*

BY

G. A. MARGULIS**

*Department of Mathematics, Yale University*
*New Haven, CT 06520, USA*
*e-mail: margulis@math.yale.edu*

AND

S. MOZES†

*Institute of Mathematics, The Hebrew University of Jerusalem*
*Givat Ram, Jerusalem 91904, Israel*
*e-mail: mozes@math.huji.ac.il*

ABSTRACT

Several aperiodic hyperbolic tiling systems consisting of a single convex tile are constructed.

The main purpose of this paper is to construct for each $n \geq 3$ a convex hyperbolic $n$-gon $P$ such that $P$ can tessellate the hyperbolic plane $\mathbb{H}^2$ but cannot tessellate any compact quotient of $\mathbb{H}^2$. We refer to [GS] for definitions and discussions of various tilings. Let $X$ be either $\mathbb{R}^d$ or $\mathbb{H}^2$. An $X$*-tiling system* is a finite collection of subsets of $X$, called "tiles", each being homeomorphic to a compact ball in $X$. A *tessellation* by these tiles is a decomposition of $X$, or more generally of a quotient of $X$, into a union of isometric copies of the various tiles intersecting only at their boundaries. A tiling system is called *aperiodic* if no compact quotient of $X$ (by a discrete group of isometries acting freely on $X$) may be tessellated

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by the tiles of the system. The first example of an aperiodic $\mathbb{R}^2$-tiling system was given by R. Berger, [Ber]. The well known example of Penrose provides an aperiodic $\mathbb{R}^2$-tiling system consisting of two tiles. Those, however, are not convex. The minimal known example, due to R. Ammann, of an aperiodic $\mathbb{R}^2$-tiling system consisting of convex polygons contains 3 polygons: a pentagon and two hexagons, cf. [GS] pages 547–549. It is an open question whether there exists an aperiodic tiling system for $\mathbb{R}^2$ consisting of a single tile. For $\mathbb{R}^3$ there is an example due to J. H. Conway, following P. Schmitt, cf. [Rad2] [Sen], of a single convex polytope which can tessellate $\mathbb{R}^3$ but not any compact 3-dimensional torus. R. Penrose gave in [Pen] an example of a single (non-convex) hyperbolic tile which can tessellate $\mathbb{H}^2$ but not any compact quotient $S = \Gamma\backslash\mathbb{H}^2$. J. Block and S. Weinberger have proved the existence of an aperiodic tiling system for any "non-amenable" space, [BW].

In both the Penrose example, [Pen], and the examples due to Block and Weinberger, the fact that the corresponding tiling systems cannot tessellate a compact quotient follows from a certain "imbalance" of the tiles. The tile constructed by Penrose has one outgoing indentation and two ingoing indentations. In any tessellation by copies of this tile the outgoing indentation of each tile must fit into an ingoing indentation of some tile. Since in a tessellation of a compact quotient one has only finitely many tiles, the number of incoming indentations is greater than that of the outgoing ones leading to an imbalance, showing that such a tessellation cannot exist. This idea plays a role also in some of the families of aperiodic polygons we shall construct. Another method of showing that a single-tile $\mathbb{H}^2$-tiling system is aperiodic is based on the following:

LEMMA 1: *A hyperbolic tiling system consisting of a single tile whose area is not a rational multiple of $\pi$ is aperiodic.*

*Proof:* By a well known corollary of the Gauss–Bonnet formula the area of compact quotient of $\mathbb{H}^2$ is an integer multiple of $\pi$. ∎

The following proposition allows us to show that certain polygons can tessellate the hyperbolic plane.

PROPOSITION 2: *Let $Q$ be a convex polygon in $\mathbb{H}^2$ having $n \geq 4$ vertices. Denote by $\alpha_i$, $1 \leq i \leq n$, its angles. Assume that:*

(1) *All the sides of $Q$ are of equal length.*
(2) *For any $i, j \in \{1, 2, \ldots, n\}$, $\alpha_i + \alpha_j \leq \pi$.*
(3) *For any $i_1, i_2, i_3 \in \{1, 2, \ldots, n\}$ there exists $r > 3$ and $i_j \in \{1, 2, \ldots, n\}$, $4 \leq j \leq r$ such that $\sum_{j=1}^{r} \alpha_{i_j} = 2\pi$.*

*(Indices appearing in (2) and (3) are not necessarily distinct.) Then there is a tessellation of $\mathbb{H}^2$ by copies of $Q$.*

*Proof:* We shall construct an increasing sequence of finite tessellations $\Omega_k$ consisting of copies of $Q$. We shall use "$\Omega_k$" to refer both to the tessellation and to the union of the tiles appearing in it. Let $\Omega_1$ consist of a single copy of $Q$. Assume by induction that:

(1) $\Omega_k$ is a compact convex polygon.
(2) Each vertex on the boundary of $\Omega_k$ belongs to at most two copies of $Q$ belonging to the tessellation forming $\Omega_k$.
(3) There exists a vertex of $\Omega_k$ belonging to a single copy of $Q$.

Observe that $\Omega_1$ satisfies these assumptions.

Let us denote the vertices along the boundary of $\Omega_k$ cyclically by $v_1, v_2, \ldots, v_m$. Without loss of generality we may assume that the vertex $v_m$ belongs to a single copy of $Q$. Consider the vertex $v_1$. Since there are at most two copies of $Q$ (both belonging to $\Omega_k$) containing it, we may, using assumptions (3) and (1) of the proposition, place at this vertex new copies of $Q$ completing the angle at this point to $2\pi$. Observe also that, since by the induction hypothesis (1) $\Omega_k$ is convex, adding these new tiles forms an admissible tessellation, i.e., there is no overlap between tiles. Next we consider the vertex $v_2$. Observe that this vertex belongs to two or three copies of $Q$. Again using the hypotheses (3) and (1) of the proposition as well as the convexity of $\Omega_k$ we can add new copies of $Q$ touching $v_2$ tessellating the whole $2\pi$ angle around it. We continue this way treating the vertices $v_i$ sequentially. Observe that at the last step we will have three copies of $Q$ containing the vertex $v_m$. The resulting tessellation is $\Omega_{k+1}$. The induction hypotheses (2) and (3) follow immediately from the construction. The convexity of $\Omega_{k+1}$ follows from assumption (2) of the proposition together with the fact that at each boundary the vertex of $\Omega_{k+1}$ belongs to at most two copies of $Q$. ∎

COROLLARY 3: *Let $Q$ be a convex hyperbolic $n$-gon ($n \geq 4$) with equal sides and angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that $\sum_{i=1}^{n} c_i\alpha_i = 2\pi$ for some integers $c_i \geq 3$ and $\alpha_j \leq \pi/2$, $1 \leq j \leq n$. Then there exists a tessellation of $\mathbb{H}^2$ by copies of $Q$.*

THEOREM 4: *For every $n \geq 3$ there exists an aperiodic tiling system whose set of tiles consists of a single convex hyperbolic $n$-gon.*

*Proof:* We shall treat separately the cases of $n = 3, 4$ and of $n \geq 5$.

$n = 3, 4$: Let $\gamma$ and $\beta$ be positive numbers such that $7\gamma + 12\beta = 2\pi$ and $\gamma \notin \mathbb{Q}\pi$. It is a well known fact of hyperbolic geometry that, given any three angles whose sum is smaller than $\pi$, there exists a triangle with these angles. Therefore there exists a hyperbolic isocceles triangle $P = \triangle ABC$ such that its angles are $\angle CAB = \gamma$, $\angle ABC = \angle BCA = \beta$. Let $Q = \square ABDC$ be the quadrangle consisting of $\triangle ABC$ together with an isometric triangle $\triangle DCB$ (see Figure 1).

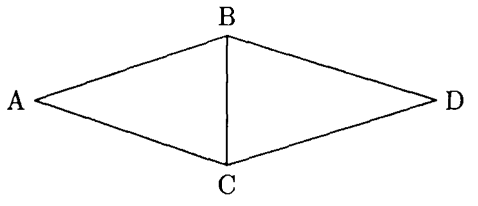


Figure 1.

The quadrangle $Q$ has equal sides and its 4 angles are $\alpha_1 = \alpha_3 = \gamma$, $\alpha_2 = \alpha_4 = 2\beta$. Observe that:

(1) $4\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 = 7\gamma + 12\beta = 2\pi$.

(2) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \notin \mathbb{Q}\pi$.

The second property follows from the choice of $\alpha_1 = \gamma \notin \mathbb{Q}\pi$ and the equality $\alpha_1 + 3\sum_{i=1}^{4} \alpha_i = 2\pi$. By the Corollary the tiling system consisting of the single quadrangle $Q$ admits a tessellation of the hyperbolic plane. By Lemma 1 it defines an aperiodic tiling system. It follows that the tiling system consisting of the triangle $\triangle ABC$ admits a tessellation of $\mathbb{H}^2$. Again by Lemma 1 it is aperiodic.

$n \geq 5$: We show now the existence of an equilateral hyperbolic $n$-gon whose angles $\alpha_i$, $1 \leq i \leq n$ satisfy:

(1) $4\alpha_1 + 3\sum_{i=2}^{n} \alpha_i = 2\pi$.

(2) $\alpha_1 \notin \mathbb{Q}\pi$.

(3) For any $1 \leq i, j \leq n$, $\alpha_i + \alpha_j \leq \pi$.

Fix some angle $\theta \notin \mathbb{Q}\pi$ such that

$$\frac{2\pi - \frac{\pi}{10}}{3n+1} < \theta < \frac{2\pi}{3n+1}.$$

Since $\theta < (n-2)\pi/n$ there exists an equilateral hyperbolic $n$-gon $Q'$ all of whose angles equal $\theta$. Denote its vertices by $A_1, A_2, A_3', A_4', A_5, \ldots, A_n$. We will deform the polygon $Q'$ to construct a new equilateral hyperbolic $n$-gon $Q$ satisfying the above conditions. To do this we fix all vertices except $A_3', A_4'$ and change these vertices in such a way that the lengths of all sides are preserved and the angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the new polygon $Q$ with vertices $A_1, A_2, A_3, A_4, \ldots, A_n$ satisfy the above properties.

Consider the quadrilateral $A_2, A_3', A_4', A_5$. It has two side lengths: $A_2A_3' = A_3'A_4' = A_4'A_5 = a$ and $A_5A_2 = b$. Let $O$ denote the center of the regular polygon with vertices $A_1, A_2, A_3', A_4', A_5, \ldots, A_n$. Consider the two isoceles triangles $\triangle A_2OA_5$ and $\triangle A_3'OA_4'$; their bases are of lengths $b$ and $a$, respectively, and all the other sides are of equal length. Thus as $\angle A_2OA_5 > \angle A_3'OA_4'$ it follows that $b > a$. Holding the vertices $A_2$ and $A_5$ fixed, we can continuously move $A_3'$ along a certain arc of the hyperbolic circle of radius $a$ around $A_2$ and $A_4'$ along an arc of the hyperbolic circle of radius $a$ around $A_5$ so that the distance between these points remains $a$. Note that since $b > a$ we can reach in this way a degenerate quadrangle in which the vertices $A_2$, $A_3'$ and $A_4'$ lie on a straight line. Note that in this degenerate quadrangle the sum of the angles is greater than $\pi$. In the original quadrangle the sum of the angle was strictly smaller than

$$4\theta < \frac{8\pi}{3n+1} < \pi - \frac{\pi}{10}.$$

Therefore at some intermediate position $A_3$, $A_4$ we have increased the sum of the angles of this quadrilateral (and hence of the polygon $Q'$) by $2\pi - (3n+1)\theta < \pi/10$. This gives a hyperbolic $n$-gon $Q$ satisfying the required properties. ∎

We remark that these aperiodic hyperbolic tiling systems consist (for $n \geq 4$) of equilateral $n$-gons. We construct next for every $n \geq 5$ a family of convex hyperbolic $n$-gons which are (generically) not equilateral, and such that each of the corresponding single-tile tiling systems admits a tessellation of $\mathbb{H}^2$ and is aperiodic. We use the upper half plane model for $\mathbb{H}^2$.

THEOREM 5: *Let $n \geq 5$ be an integer. For any positive real number $a$ let $P_a$ be the convex hyperbolic polygon whose vertices are $A_j = (j-1)a + i$, $1 \leq j \leq n-2$, $A_{n-1} = (n-3)a + (n-3)i$ and $A_n = (n-3)i$. For every $a > 0$ there exists a tessellation of $\mathbb{H}^2$ by copies of $P_a$. There exists $c_n \geq 0$ such that for almost every $a > 0$ and for every $0 < a < c_n$ the corresponding single-tile tiling system is aperiodic.*

*Proof:* Let $\Gamma_a$ be the (discrete) semigroup of isometries of $\mathbb{H}^2$ generated by the transformations

$$Tz = \frac{1}{n-3}z, \quad S_a z = z + (n-3)a \quad \text{and} \quad S_a^{-1} z = z - (n-3)a.$$

It is easily checked that the collection of translates $\{T^{-k}\gamma P_a | \ \gamma \in \Gamma_a, k \geq 0\}$ forms a tessellation of the hyperbolic plane by copies of $P_a$. Considering the dependency of the angles of $P_a$ on $a$ it follows that the area of $P_a$ is a continuous strictly increasing function of $a$. Thus for almost every $a > 0$ the area of $P_a$ is not a rational multiple of $\pi$, hence by Lemma 1 the corresponding tiling system is aperiodic. To show that for every sufficiently small $a > 0$ the corresponding tiling system is aperiodic, we color the edge $A_n A_{n-1}$ blue and each of the edges $A_j A_{j+1}$ for $1 \leq j \leq n-3$ red. Thus the polygon $P_a$ has $n-2$ colored edges of length $x$ and 2 non-colored edges both of length $y = \log(n-3)$. Observe that as $a$ tends to 0 we have:

(1) The side length $x$ tends to 0, thus for sufficiently small $a$, $x < y/5$.
(2) The angle at each of the vertices $A_j$, $2 \leq j \leq n-3$ approaches $\pi$, and the angles at the other four vertices approach $\pi/2$.

By examining the ways the various angles of $P_a$ may be combined to sum to $\pi$ or $2\pi$, one can verify that in any tessellation by copies of $P_a$:

(1) Each copy of a vertex $A_j$, $2 \leq j \leq n-3$ cannot meet the interior of an edge.
(2) A copy of a colored edge must meet a copy of a colored edge.
(3) A colored edge must meet an edge of the other color.

Now we can apply the same "imbalance" argument as in [Pen] to conclude that $P_a$ cannot tessellate any compact quotient of $\mathbb{H}^2$. Indeed, in such a tessellation we have only finitely many copies of $P_a$. Thus we must have the same number of blue edges, and of red edges, contradicting the fact that each tile has more red edges. ∎

The following theorem provides another criteria for showing that a given tiling system cannot tessellate a compact quotient of $\mathbb{H}^2$. This may be applied to establish the aperiodicity of the tiling systems of Theorem 5 as well as that of the tiling systems corresponding to the quadrangles constructed in the proof of Theorem 4.

THEOREM 6: *Let $\mathcal{T}$ be a tiling system consisting of finitely many convex polygons $\{P_i\}$. Let $\beta_1, \beta_2, \ldots, \beta_k$ be the distinct angles appearing in tiles of $\mathcal{T}$. Associate to each of the polygons $P_i$ a vector*

$$f_i = \left(\frac{x_1^i}{s_i}, \frac{x_2^i}{s_i}, \ldots, \frac{x_k^i}{s_i}\right) \in \mathbb{R}^k$$

*where $x_j^i$ is the number of angles of $P_i$ which are equal to $\beta_j$ and $s_i$ is the number of vertices of $P_i$. Let $F \subset \mathbb{R}^k$ be the convex hull of these vectors. Consider the set*

$$C = \left\{\left(\frac{c_1}{s}, \frac{c_2}{s}, \ldots, \frac{c_k}{s}\right) \mid c_j \in \mathbb{Z}^+, \sum_{j=1}^k c_j \beta_j = 2\pi \text{ or } \pi, \text{ and } s = \sum_{j=1}^k c_j\right\}.$$

*If the convex hull of $C$ is disjoint from $F$ then $\mathcal{T}$ does not admit a tessellation of any compact quotient of $\mathbb{H}^2$.*

*Proof:* Suppose there exists a tessellation of some compact quotient of $\mathbb{H}^2$ by tiles of $\mathcal{T}$. Calculating the relative frequency of each of the angles $\beta_j$, $1 \leq j \leq k$, in such a tessellation in two ways shows that the two convex sets $F$ and $\operatorname{conv}(C)$ must intersect. ∎

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## References

[Ber] R. Berger, *The undecidability of the domino problem*, Memoirs of the American Mathematical Society **66** (1966).

[BW] J. Block and S. Weinberger, *Aperiodic tilings, positive scalar curvature and amenability of spaces*, Journal of the American Mathematical Society **5** (1992), 907–918.

[GS] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman and Co., New York, 1987.

[M] S. Mozes, *Aperiodic tilings*, Inventiones Mathematicae **128** (1997), 603–611.

[Pen] R. Penrose, *Pentaplexity*, Mathematical Intelligencer **2** (1979), 32–37.

[Rad1] C. Radin, *Global order from local sources*, Bulletin of the American Mathematical Society **25** (1991), 335–364.

[Rad2] C. Radin, *Aperiodic tilings in higher dimensions*, Proceedings of the American Mathematical Society **123** (1995), 3543–3548.

[Sen] M. Senechal, *Quasicrystals and Geometry*, Cambridge University Press, 1995.